

Haskell's answer does a great job of outlining conditions that a derivative f' must satisfy, which then limits us in our search for an example. From there we see the key question: can we provide a concrete example of an everywhere differentiable function whose derivative is discontinuous on a dense, full-measure set of \mathbb{R} ? Here's a closer look at the Volterra-type functions referred to in Haskell's answer, together with a little indication as to how it might be extended.

Basic example

The basic example of a differentiable function with discontinuous derivative is

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

The differentiation rules show that this function is differentiable away from the origin and the difference quotient can be used to show that it is differentiable at the origin with value $f'(0) = 0$. A graph is illuminating as well as it shows how $\pm x^2$ forms an envelope for the function forcing differentiability.

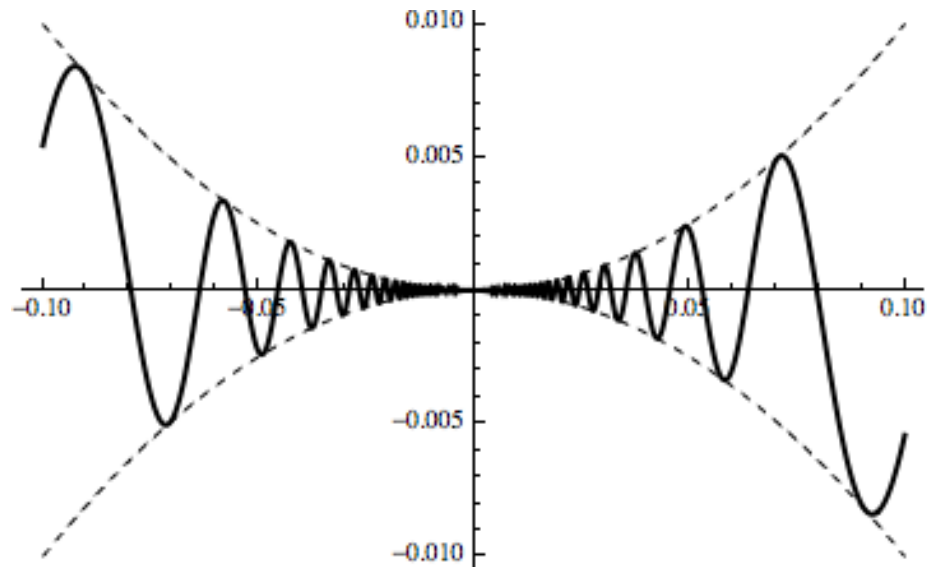


Figure 1: enter image description here

The derivative of f is

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

which is discontinuous at $x = 0$. Its graph looks something like so

Two points

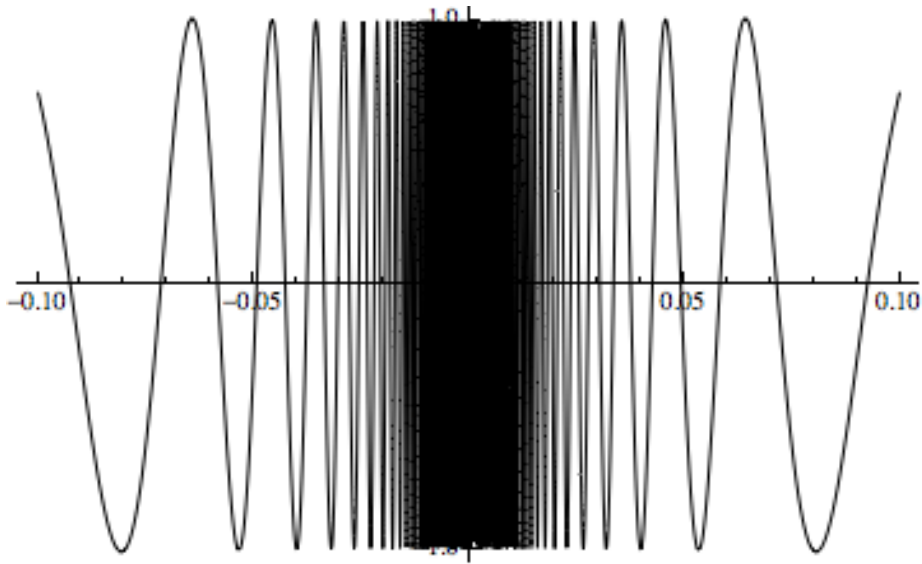


Figure 2: enter image description here

The next step is to modify this example to obtain a function that is everywhere differentiable with a derivative that is continuous on all of \mathbb{R} , except for two points. To this end, consider

$$f(x) = \begin{cases} x^2(1-x)^2 \sin\left(\frac{1}{\pi x(1-x)}\right) & \text{if } 0 < x < 1 \\ 0 & \text{else.} \end{cases}$$

The graph of f and its derivative look like so.

A cantor set of discontinuities

Now that we have a way to construct a differentiable function whose derivative is discontinuous exactly at the endpoints of an interval, it should be clear how to construct a differentiable function whose derivative is discontinuous on a Cantor set constructed in the interval. For $n \in \mathbb{N}$ and $m = 1, 2, \dots, 2^n$, let $I_{m,n}$ denote one of the 2^n intervals removed during the n^{th} stage of construction of the Cantor set. Then let $f_{m,n}$ be scaled to have support $I_{m,n}$ and to have maximum value 4^{-n} . The function

$$F(x) = \sum_{n=0}^{\infty} \sum_{m=1}^{2^n} f_{m,n}(x)$$

will be everywhere differentiable but its derivative will be discontinuous on the given Cantor set. Assuming we do this with Cantor's standard ternary set, we get a picture that looks something like so:

Of course, there's really a sequence of functions here and care needs to be taken

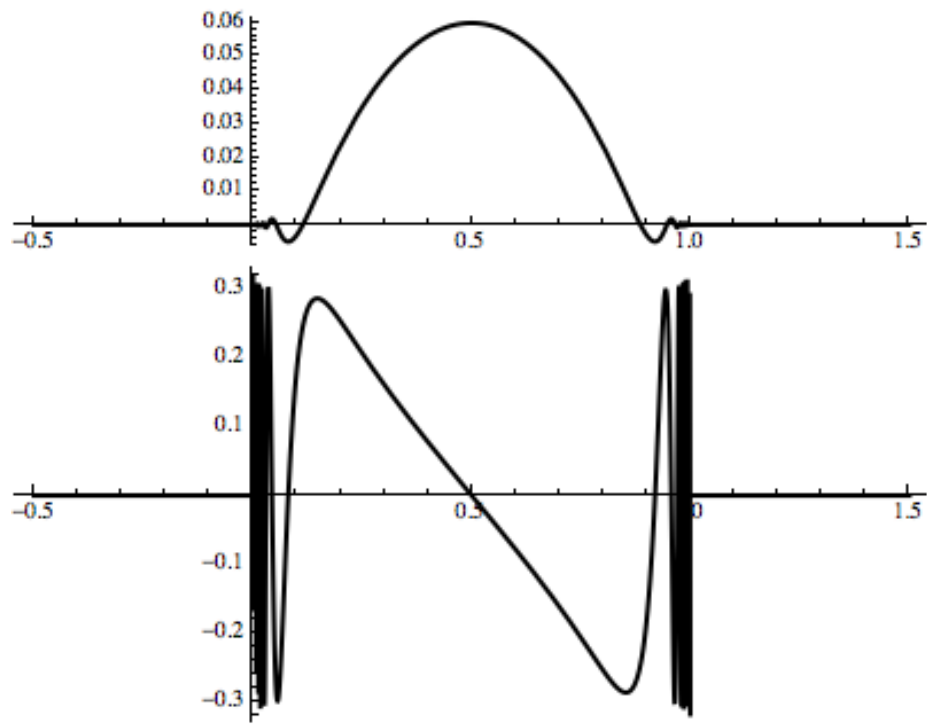


Figure 3: enter image description here

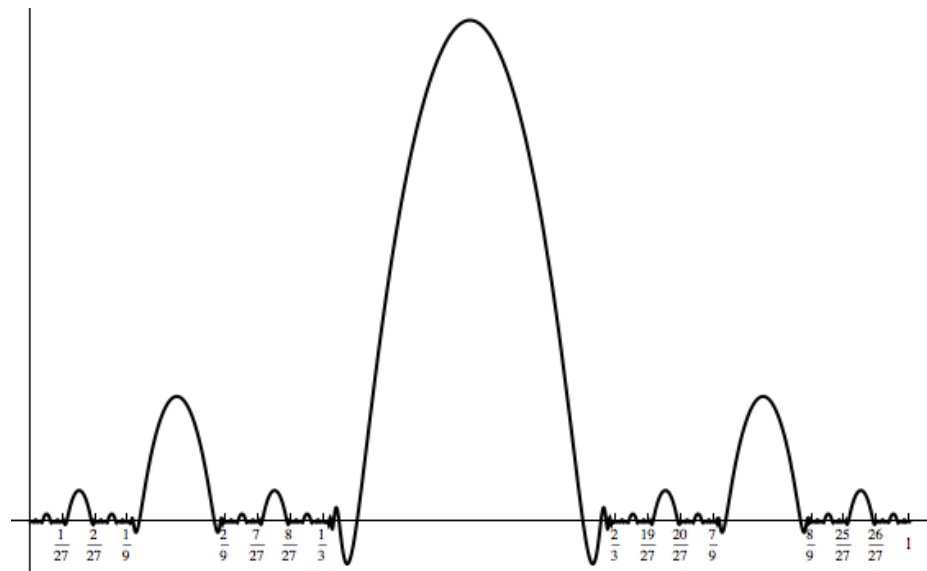


Figure 4: enter image description here

to show that the limit is truly differentiable. Let

$$F_N(x) = \sum_{n=1}^N \sum_{m=1}^{2^n} f_{m,n}(x).$$

The standard theorem then states that, as long as F_N converges and F'_N converges uniformly, then the limit of $F_N(x)$ will be differentiable. This is guaranteed by the choice of 4^{-n} as the max for $f_{m,n}$.

Increasing the measure

Again, the last example refers to the standard Cantor ternary set but there's no reason this can't be done with *any* Cantor set. In particular, it can be done with a so-called [fat Cantor set](#), which can have positive measure arbitrarily close to the measure of the interval containing it. We immediately produce an everywhere differentiable function whose derivative is discontinuous on a nowhere dense set of positive measure. (Of course, care must again be taken to scale the heights of the functions go to zero quickly enough to guarantee differentiability.)

Finally, we can fill the holes of the removed intervals with more Cantor sets (and their corresponding functions) in such a way that the union of all of them is of full measure. This allows us to construct an everywhere differentiable function with derivative that is discontinuous on the union of those Cantor sets, which is a set of full measure.